

The dynamic centre of gravity

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We often read that in linear gravity, the centre of gravity and the centre of mass are the same. Note that there is no such thing as linear gravity, since the non linear inverse square law applies to all gravitational fields. However, for many practical applications, it makes sense to approximate the gravitational field as being linear, that is, we can neglect the difference that exists in the weight of masses as they are moved further or closer to the second body. However, even though the centre of gravity is sometimes erroneously defined as totally equivalent to the centre of mass, these two terms are two completely different things, with important consequences and important uses in space and other advanced research.

Centre of gravity of a rod having uniform cross section and density.

For linear gravity

Centre of gravity (Y_{CG}) will be at the same location as the centre of mass (Y_{CM}), in which case:

$$Y_{CM} \times \sum m = \sum m * y$$

$$Y_{CM} = \int_0^L y dy / \int_0^L dy$$

$$Y_{CM} = [y^2/2y]_0^L$$

$Y_{CM} = L/2$... that is, both Y_{CM} and Y_{CG} are located exactly at its mid point, and are independent of the rods orientation.

It is very important to understand that even though in such a case, the CG and CM are located at the same point, they are still two different parameters, describing two different things. In the following example you will better appreciate the difference between them. To resolve the CG in 3D, simply use the same method applied to the X and Z axis.

For non linear gravity

Whilst the CM is a fixed point, the CG is dynamic as it depends on the instantaneous orientation of the object with respect to the gravitational field. While CM and CG equate to the same point in an approximated uniform gravitational field, this is not true when one takes into account the inverse square law variation of any real gravitational field, which should strictly speaking be always taken into account when calculating the CG and can result in small generation of a moment if the rod is in a frictionless environment or in space.

Let R be the distance from the centre of the earth to the lowest point on the rod in its vertical position. Let L be the length of the rod, Y_{CG} be the cg point measured from the lower end of the rod, and 'dy' representing a small slice of the rod. Since the rod has uniform radius and density, such parameters all cancel out, and we are left with:

$$\int_0^L dy / (R+Y_{CG})^2 = \int_R^{R+L} dy/y^2$$

$$L / (R+Y_{CG})^2 = [-1/y]_R^{R+L}$$

$$L / (R+Y_{CG})^2 = [1/R - 1/(R+L)]$$

$$Y_{CG} = \left\{ \sqrt{ [L / (1/R - 1/(R+L))] } \right\} - R$$

At sea level $R=6.4E6m$. Let's calculate the CG of a rod 10m long in its vertical position.

$$Y_{CG} = \left\{ \sqrt{ [L / (1/R - 1/(R+L))] } \right\} - R, \text{ where } R= 6.4E6m \text{ and } L=10m$$

$Y_{CG} = 4.99999805m$, whilst Y_{CM} is always at the midpoint, that is at 5m.

As expected, the dynamic CG of a long rod in its vertical position is lower than its fixed centre of mass, since the lowest parts of its mass contribute slightly more to its weight than the upper parts, due to the inverse square law of gravity.

Gravity gradient importance in space

This dynamic location of the centre of gravity is what tends to rotate satellites in space, aligning their long axis towards the direction of earth, the stable position being the one with the rod's CG on the side facing the earth. This effect is usually referred to as the gravity gradient effect. The rod will move its dynamic CG on the other half if rotated by 180 degrees from any one of its two stable positions. Any other orientation will generate a torque in a way to reduce the rods potential energy. This effect is also utilised in space gravity gradient booms, such as those used by NASA in LEO satellites as early as 1960's. In such units, a separate mass tip is deployed from the satellite and held at a large distance away from the satellite body. Since the two bodies are so far from each other, and the gravitational field is stronger closest to the earth, the CG of the whole system moves slightly towards the earth, and also becomes the new point of orbit of the system. Thus the satellite no longer orbits along its centre of mass. Such a system attains its equilibrium when the satellite, tip mass and earth are all in line.